Midterm Examination 2
ECE 301

Division 2, Fall 2006

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Instructions:

1. Wait for the "BEGIN" signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.

2. You have 50 minutes to complete the 5 questions contained in this exam, for a total of up to 105 points. **When the end of the exam is announced, you must stop writing immediately**. Anyone caught writing after the exam is over will get a grade of zero.

3. This booklet contains 13 pages. The last three pages contain a table of formulas and properties. You may tear out these three pages **once the exam begins**.

4. This is a closed book exam. The use of calculators is prohibited. Cell phones, pagers, and all other electronic communication devices as well as i-pods are strictly forbidden.

Name:______________________________

Email:______________________________

Signature:______________________________

(15 pts) **1.** Compute the Fourier transform of the DT signal

$$x[n] = n^2 u[n-2] - n^2 u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

(15 pts) **2.** Show that the Fourier transform of the CT signal $x(t) = cos(\omega_0 t)$ is $\mathcal{X}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

(15 pts) **3.** Given is a DT signal $x[n] = \frac{1}{g[n]^2}$ where $g[n]$ is a pure imaginary signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{cos\omega}$. Explain why Bob's answer is wrong.

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{sin\omega}$. Could Alice be right? Explain.

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{\omega^2}$. Could Devin be right? Explain.

4. A discrete-time LTI system has frequency response

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$?

(15 pts) b) Find the unit impulse response of this system.

(20 pts) **5.** Use the definition of the Fourier transform (*not* the properties listed in the table) to prove the following Fourier transform property.

$$x(at+b) \xrightarrow{F} \frac{e^{j\omega\frac{b}{a}}}{-a}\mathcal{X}\left(\frac{\omega}{a}\right) \text{ for any } a, b \text{ real numbers with } a < 0.$$

Table

1 Definition of the Continuous-time Fourier Transform

Let $x(t)$ be a signal and denote by $\mathcal{X}(\omega)$ its Fourier transform.

$$\text{Fourier Transform: } \mathcal{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{1}$$

$$\text{Inverse Fourier Transform: } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{X}(\omega)e^{j\omega t}d\omega \tag{2}$$

2 Properties of the Continuous-time Fourier Transform

Let $x(t)$ be a continuous-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another continuous-time signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

$$\text{Linearity: } ax(t) + by(t) \xrightarrow{\mathcal{F}} a\mathcal{X}(\omega) + b\mathcal{Y}(\omega) \tag{3}$$

$$\text{Time Shifting: } x(t - t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0}\mathcal{X}(\omega) \tag{4}$$

$$\text{Frequency Shifting: } e^{j\omega_0 t}x(t) \xrightarrow{\mathcal{F}} \mathcal{X}(\omega - \omega_0) \tag{5}$$

$$\text{Conjugation: } x^*(t) \xrightarrow{\mathcal{F}} \mathcal{X}^*(-\omega) \tag{6}$$

$$\text{Scaling: } x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|}\mathcal{X}\left(\frac{\omega}{a}\right) \tag{7}$$

$$\text{Multiplication: } x(t)y(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi}\mathcal{X}(\omega) * \mathcal{Y}(\omega) \tag{8}$$

$$\text{Convolution: } x(t) * y(t) \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\mathcal{Y}(\omega) \tag{9}$$

$$\text{Differentiation in Time: } \frac{d}{dt}x(t) \xrightarrow{\mathcal{F}} j\omega\mathcal{X}(\omega) \tag{10}$$

$$x(t)\text{real} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega) = \mathcal{X}^*(-\omega) \tag{11}$$

$$x(t)\text{real and even} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{real and even} \tag{12}$$

$$x(t)\text{real and odd} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{pure imaginary and odd} \tag{13}$$

$$\text{Parseval's Relation: } \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{X}(\omega)|^2 \, d\omega \tag{14}$$

3 Some Continuous-time Fourier Transforms

$$\delta(t) \xrightarrow{\mathcal{F}} 1 \tag{15}$$

$$e^{-at}u(t), \mathcal{R}e\{a\} > 0 \xrightarrow{\mathcal{F}} \frac{1}{a+j\omega} \tag{16}$$

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \xrightarrow{\mathcal{F}} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}) \tag{17}$$

4 Fourier Series of Continuous-time Periodic Signals with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \tag{18}$$

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt \tag{19}$$

5 Fourier Series of Discrete-time Periodic Signals with period N

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{20}$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n} \tag{21}$$

6 Definition of the Discrete-time Fourier Transform

Let $x[n]$ be a discrete-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform.

$$\text{Fourier Transform: } \mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{22}$$

$$\text{Inverse Fourier Transform: } x[n] = \frac{1}{2\pi}\int_{2\pi} \mathcal{X}(\omega)e^{j\omega n}d\omega \tag{23}$$

7 Properties of the Discrete-time Fourier Transform

Let $x[n]$ and $y[n]$ be DT signals. Denote by $\mathcal{X}(\omega)$ and $\mathcal{Y}(\omega)$ their Fourier transforms.

$$\text{Linearity: } ax[n]+by[n] \xrightarrow{\mathcal{F}} a\mathcal{X}(\omega)+b\mathcal{Y}(\omega) \quad (24)$$

$$\text{Time Shifting: } x[n-n_0] \xrightarrow{\mathcal{F}} e^{-j\omega n_0}\mathcal{X}(\omega) \quad (25)$$

$$\text{Frequency Shifting: } e^{j\omega_0 n}x[n] \xrightarrow{\mathcal{F}} \mathcal{X}(\omega-\omega_0) \quad (26)$$

$$\text{Conjugation: } x^*[n] \xrightarrow{\mathcal{F}} \mathcal{X}^*(-\omega) \quad (27)$$

$$\text{Time Reversal: } x[-n] \xrightarrow{\mathcal{F}} \mathcal{X}(-\omega) \quad (28)$$

$$x_k[n] = \begin{cases} x[\frac{n}{k}], & \text{if } k \text{ divides } n \\ 0, & \text{else.} \end{cases} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega) \quad (29)$$

$$\text{Multiplication: } x[n]y[n] \xrightarrow{\mathcal{F}} \frac{1}{2\pi}\mathcal{X}(\omega)*\mathcal{Y}(\omega) \quad (30)$$

$$\text{Convolution: } x(t)*y(t) \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\mathcal{Y}(\omega) \quad (31)$$

$$\text{Differentiation: } x[n]-x[n-1] \xrightarrow{\mathcal{F}} (1-e^{-j\omega})\mathcal{X}(\omega) \quad (32)$$

$$\text{Accumulation: } \sum_{k=-\infty}^{n} x[k] \xrightarrow{\mathcal{F}} \frac{\mathcal{X}(\omega)}{1-e^{-j\omega}} + \pi\mathcal{X}(0)\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k) \quad (33)$$

$$x[n]\text{real} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)=\mathcal{X}^*(-\omega) \quad (34)$$

$$x[n]\text{real and even} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{real and even} \quad (35)$$

$$x[n]\text{real and odd} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{pure imaginary and odd} \quad (36)$$

$$\text{Parseval's Relation: } \sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|\mathcal{X}(w)|^2\,d\omega \quad (37)$$

8 Some Discrete-time Fourier Transforms

$$\sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n} \xrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \quad (38)$$

$$1 \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \quad (39)$$

$$\frac{\sin Wn}{\pi n}, 0 < W < \pi \xrightarrow{\mathcal{F}} \mathcal{X}(\omega) = \begin{cases} 1, & 0 \leq |\omega| < W \\ 0, & \pi \geq |\omega| > W \end{cases} \quad (40)$$

$$\mathcal{X}(\omega) \text{periodic with period } 2\pi$$

$$\delta[n] \xrightarrow{\mathcal{F}} 1 \quad (41)$$

$$\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{1 - \alpha e^{-j\omega}} \quad (42)$$

$$(n+1)\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{(1 - \alpha e^{-j\omega})^2} \quad (43)$$