E302: HOMEWORK-5

(1)(a) $f_{X,Y}(x,y) = K(x+2y) \quad 0<x<1,\ 0<y<1$

To find K: $\int_0^1\int_0^1 K(x+2y)\,dx\,dy = 1$

Solving: $K = \frac{2}{3}$

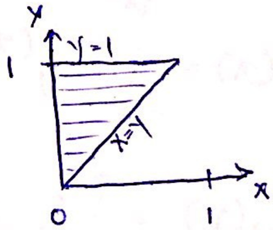
(b) Marginal PDF's:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 \frac{2}{3}(x+2y)\,dy = \frac{2}{3}(x+1)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 \frac{2}{3}(x+2y)\,dx = \frac{1}{3} + \frac{4}{3}y$$

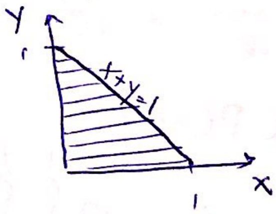
(c) Find $P(X<Y)$

$$P(X<Y) = \int_0^1\int_0^y \frac{2}{3}(x+2y)\,dx\,dy = \frac{5}{9}$$



(d) Find $P(X+Y\le 1)$

$$P(X+Y\le 1) = \frac{2}{3}\int_0^1\int_0^{1-y} (x+2y)\,dx\,dy = \frac{1}{3}$$



(e) Mean values of X & Y

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2}{3}x(x+2y)\,dx\,dy = \frac{5}{9}$$

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2}{3}y(x+2y)\,dx\,dy = \frac{11}{18}$$

(f) Correlation: $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy = \frac{1}{3}$

Covariance: $E\left[\left(X-\frac{5}{9}\right)\left(Y-\frac{11}{18}\right)\right] = E\left[XY - \frac{11}{18}X - \frac{5}{9}Y + \frac{55}{162}\right]$

$$= \frac{-1}{162}$$

(g) X & Y are not uncorrelated as $Cov[X,Y] \neq 0$

X & Y are not orthogonal as $E[XY] \neq 0$

X & Y are not independent as ~~not~~ $f_{XY}(x,y) \neq f_X(x) f_Y(y)$

(2) (a) Find K: $\int_0^1 \int_0^1 (Kxy - Kx^2y)\,dx\,dy = 1$

Solving $K = 12$

(b) Find the marginal PDF's:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 (12xy - 12x^2y)\,dy = 6x - 6x^2$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 (12xy - 12x^2y)\,dx = 2y$$

(c) Find $P(X<Y)$

$$P(X<Y) = 12\int_0^1 \int_0^y (xy - x^2y)\,dx\,dy = 7/10$$

(d) Find $P(X+Y \leq 1)$

$$P(X+Y \leq 1) = 12\int_0^1 \int_0^{1-y} (xy - x^2y)\,dx\,dy = \frac{3}{10}$$

(e) Mean values of X and Y:

$$E[X] = 12\int_0^1 \int_0^1 (x^2y - x^3y)\,dx\,dy = \frac{1}{2}$$

$$E[Y] = 12\int_0^1 \int_0^1 (xy^2 - x^2y^2)\,dx\,dy = \frac{2}{3}$$

(f) Correlation : $E[XY] = 12\int_0^1\int_0^1 (x^2y^2 - x^3y^2)\,dx\,dy = \frac{1}{3}$

$Cov[XY] = E\left[(X-\frac{1}{2})(Y-\frac{2}{3})\right] = E\left[XY - \frac{2}{3}X - \frac{1}{12}Y + \frac{1}{3}\right] = 0$

(g) X & Y are not orthogonal as $E[XY] \neq 0$

X & Y are uncorrelated as $Cov[X,Y] = 0$

X & Y are independent as $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

(3) (a) $f_{Y/X}(y/x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{\frac{2}{3}(x+2y)}{\frac{2}{3}(x+1)} = \dfrac{x+2y}{x+1}$

(b) Find $P(Y>X|X=x)$ -

$P(Y>X|X=x) = \int_x^1 f_{Y/X}(y|x)\,dy = \int_x^1 \frac{x+2y}{x+1}\,dy = \frac{-2x^2+x+1}{x+1}$

(c) $P(Y>X) = \int_{-\infty}^{\infty} P(Y>X|X=x) f_X(x)\,dx$

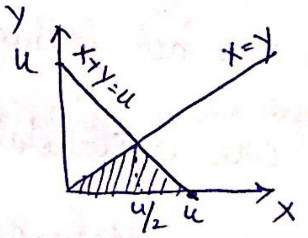
$= \int_0^1 \frac{-2x^2+x+1}{x+1}\left(\frac{2}{3}(x+1)\right)dx = \frac{5}{9}$

(d) $E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y/X}(y|x)\,dy = \frac{1}{x+1}\left[\frac{x}{2} + \frac{2}{3}\right]$

that y has to be less than x by the given bounds.

(b) Find k: $\int_0^\infty \int_y^\infty k e^{-(x+y)} \, dx \, dy = 1$, solving $k=2$

(c) $U = X + Y$

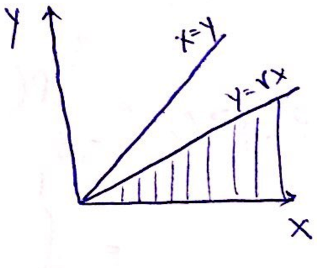


$$P(U \le u) = \int_0^{x} \int_0^{u/2} 2e^{-(x+y)} \, dx \, dy + \int_{u/2}^{u} \int_0^{u-x} 2e^{-(x+y)} \, dy \, dx$$

$$= \int_0^{u/2} \int_y^{u-y} 2e^{-(x+y)} \, dx \, dy = -ue^{-u} - e^{-u} + 1$$

$$F_U(u) = -ue^{-u} - e^{-u} + 1 \Rightarrow f_U(u) = \frac{d}{du} F_U(u) = ue^{-u}$$

(d) $V = Y/X$



$$P(V \le v) = P(Y/X \le v) = P(Y \le vX)$$

$$= \int_0^\infty \int_{y/v}^\infty 2e^{-(x+y)} \, dx \, dy = 2 - \frac{2}{v+1}$$

$$f_V(v) = \frac{d}{dv} F_V(v) = \frac{2}{(v+1)^2} \quad \text{for } v \le 1$$

$$\frac{1}{2} \quad \text{for } v > 1$$

(5)(a) $P(\sqrt{X^2+Y^2} \geq r) = 1 - P(\sqrt{X^2+Y^2} < r)$

$$= 1 - \iint_{x^2+y^2 \leq r} f_{X,Y}(x,y)\,dx\,dy$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x^2+y^2)}{2\sigma^2}\right)$$ as $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = \sigma$

$$\therefore P(\sqrt{X^2+Y^2} \geq r) = 1 - \iint_{x^2+y^2 \leq r} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x^2+y^2)}{2\sigma^2}\right) dx\,dy$$

$$\boxed{\begin{array}{l} r = \sqrt{x^2+y^2} \\ x = r\cos\theta \\ y = r\sin\theta \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{array}}$$

$$= 1 - \frac{1}{2\pi\sigma^2} \int_0^{2\pi}\int_0^{R} \exp\left(\frac{-r^2}{2\sigma^2}\right) r\,dr\,d\theta$$

$$= e^{-R^2/2\sigma^2}$$

(b) $F_R(r) = P(R \leq r) = 1 - e^{-r^2/2\sigma^2}$

$$f_R(r) = \frac{d}{dr} F_R(r) = \frac{2r}{2\sigma^2} e^{-r^2/2\sigma^2}$$

$$= \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$$ a Cauchy distribution