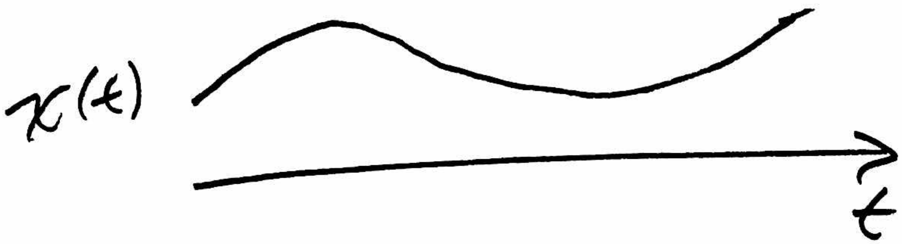
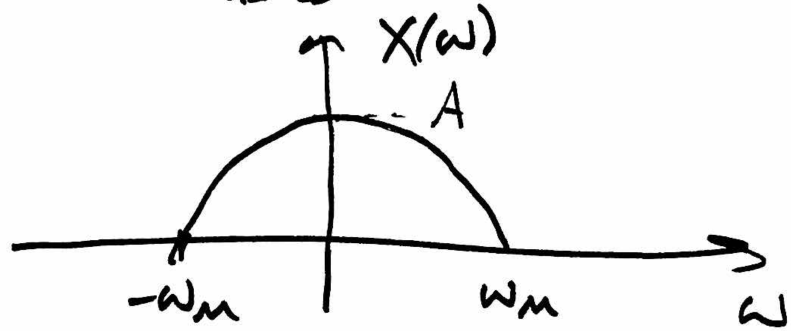
Discrete Time Fourier Transform (DTFT)

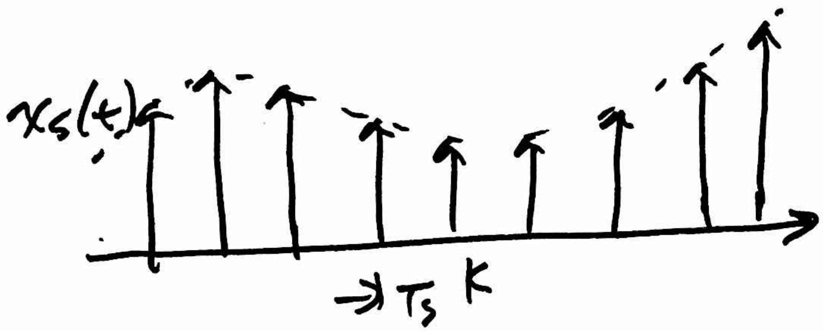
→ Going from sampling

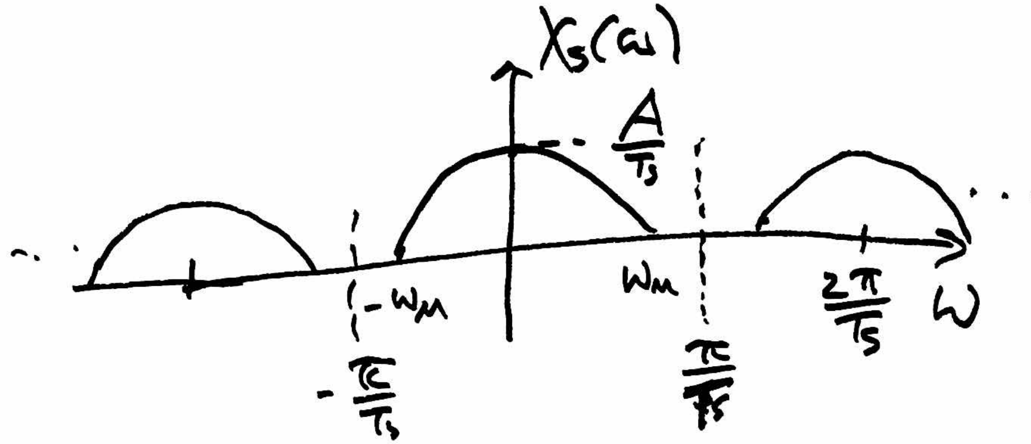
- Let $x(t)$ be a bandlimited signal ($X(\omega)=0 \quad \omega>\omega_M$)

and $x_s(t) = x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = x_s(t) = \sum_{n=-\infty}^{\infty}x(nT_s)\,\delta(t-nT_s)$









Call $x(nT_s) = x[n]$

Look at the synthesis for $x[n]$:

$$x[n] = x(nT_s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{+j\omega n T_s} d\omega = \frac{1}{2\pi}\int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} X_s(\omega)\cdot T_s e^{+j\omega n T_s} d\omega$$

In order to "untie" the dependence on time, we'll change units:

ω : $\frac{\text{rad}}{\text{sec}}$ $\qquad \phi$: $\frac{\text{rad}}{\text{sample}}$

$$\frac{\text{rad}}{\text{sec}} \,\Big|\, \frac{T_s \text{ sec}}{\text{sample}} \Rightarrow \omega T_s = \phi \qquad \omega = \frac{\phi}{T_s}$$

$$d\omega = \frac{1}{T_s} d\phi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_s\left(\frac{\phi}{T_s}\right) T_s e^{+j\left(\frac{\phi}{T_s}\right) n T_s} \frac{1}{T_s} d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_s\left(\frac{\phi}{T_s}\right) e^{+j\phi n} d\phi$$ $\Rightarrow$ No longer depends on seconds, just samples.

Looking at the analysis equation:

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)\right)e^{-j\omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\left(\int_{-\infty}^{\infty}\delta(t-nT_s)e^{-j\omega t}dt\right)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s}\overbrace{\int_{-\infty}^{\infty}\delta(t-nT_s)dt}^{1}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT_s}$$

Normalize frequency: $\omega T_s = \phi$ $\quad \omega = \frac{\phi}{T_s}$

$$X_s\left(\frac{\phi}{T_s}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\phi n}$$

From now on, we will use $X(\omega)$ when talking about the DTFT, not $X_s\left(\frac{\phi}{T_s}\right)$. Context will tell whether ω is $\frac{\text{rad}}{\text{sec}}$ or $\frac{\text{rad}}{\text{sample}}$.

$$\underbrace{\frac{\text{rad}}{\text{sec}}}_{\substack{\text{CTFT} \\ x(t)}} \qquad \underbrace{\frac{\text{rad}}{\text{sample}}}_{\substack{\text{DTFT} \\ x[n]}}$$

$$X_s\left(\frac{\phi}{T_s}\right) \longrightarrow X(\omega)$$

(_not_ the same $X(\omega)$ from page 1)

DTFT equations

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega + 2\pi l) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega+2\pi l)n}$$

$$= \sum_n x[n] e^{-j\omega n} \cancelto{1}{e^{-j2\pi l n}}$$

$$= \sum_n x[n] e^{-j\omega n}$$

■ Linearity & Time shift

$\rightarrow \text{DTFT}\{a x[n] + b y[n]\} = a\,\text{DTFT}\{x[n]\} + b\,\text{DTFT}\{y[n]\}$

$\rightarrow x[n] \xleftrightarrow{\text{DTFT}} X(\omega)$

$x[n-n_0] \xleftrightarrow{\text{DTFT}} e^{-j\omega n_0} X(\omega)$

Ex

$$y[n] - 0.5\,y[n-1] = x[n] + 2x[n-1]$$

Find $H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$

$y[n] \xleftrightarrow{\text{DTFT}} Y(\omega)$

$x[n] \xleftrightarrow{\text{DTFT}} X(\omega)$

$$Y(\omega) - 0.5\,Y(\omega)e^{-j\omega} = X(\omega) + 2X(\omega)e^{-j\omega}$$

$$Y(\omega)\left(1 - 0.5e^{-j\omega}\right) = X(\omega)\left(1 + 2e^{-j\omega}\right)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 + 2e^{-j\omega}}{1 - 0.5e^{-j\omega}}$$ $\rightarrow$ frequency response of the system

Ex $x[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{else} \end{cases}$

Find $X(\omega)$:

Apply the DTFT analysis eqn:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

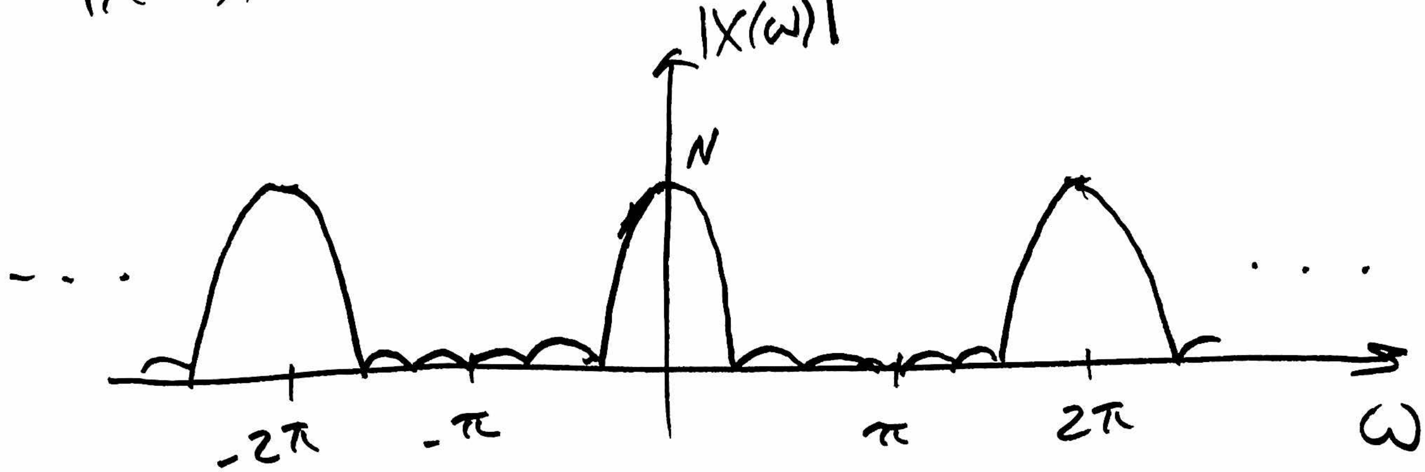
$$= \sum_{n=0}^{N-1} 1 \cdot e^{-j\omega n} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\frac{\omega N}{2}}}{e^{-j\frac{\omega}{2}}} \frac{e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}$$

$$= e^{-j\frac{\omega}{2}(N-1)} \underbrace{\frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}}$$

Looks like a periodic sinc.
Sometimes called psinc

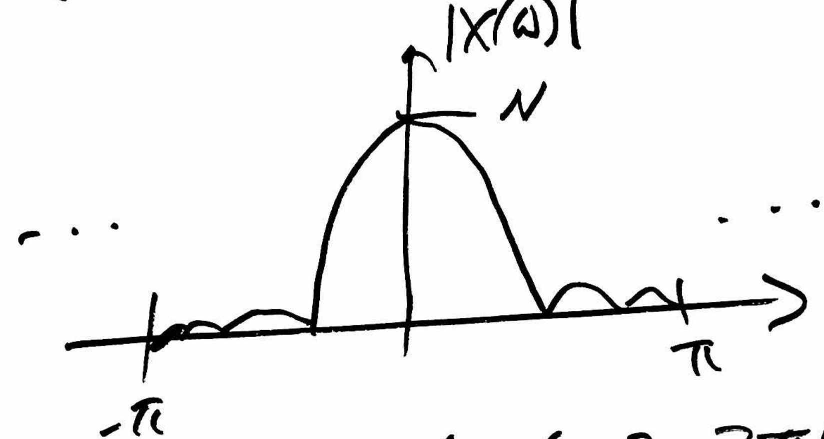
Plot $|X(\omega)|$



$\sin(\frac{\omega}{2}) = 0$ when $\frac{\omega}{2} = \pi k$ $\quad \omega = 2\pi k$

$\sin(\frac{\omega}{2}N) = 0$ when $\frac{\omega}{2}N = \pi l$ $\quad \omega = \frac{2\pi l}{N}$

Generally, we'll plot & define $X(\omega)$ (DTFT) from $-\pi$ to π



periodic with period 2π

"Low" frequencies are centered on $2\pi k$, "high" on $\pi + 2\pi k$.