Real Analysis Qual Prep — Summer 2009

Assignment 6: Absolute Continuity and Differentiation

1. Construct absolutely continuous functions $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbf{R}$ such that $g \circ f$ is not AC.

2. Suppose $f \in L^1([a, b])$ and

   $$\int_a^c f(t)dt = 0 \text{ for all } c \in [a, b].$$

   Show that $f = 0$ a.e.

3. Does there exist a strictly increasing function f defined on an interval I such that $f' = 0$ a.e. on I?

4. Let I and J be bounded intervals, $f : I \to J$ AC and $\phi : J \to \mathbf{R}$ Lipschitz. Show that $\phi \circ f$ is AC on I.

5. Let $f : [0, 1] \to \mathbf{R}$ be C^1, and let

   $$Z = \{x \in [0, 1] \, : \, f'(x) = 0\}.$$

   Show that $\mu\{f(Z)\} = 0$.

6. Let $g : [0, 1] \to \mathbf{R}$ be continuous, $g(y) \leq g(x)$ if $y \leq x$.
   (a) Suppose $g'(x) = 0$ for all but perhaps finitely many x. Show that g is constant.
   (b) Suppose $g'(x) = 0$ for all but perhaps countably many x. Show that g is constant.

7. True or false: There exists a measurable set E satisfying $|E \cap I| = c|I|$, $0 < c < 1$ for all bounded intervals I.

8. Let $\{f_n\} \in \mathrm{AC}(I)$, $I = [a, b]$. Assume $f_n \to f$ in L^1 and $\{f_n'\}$ is Cauchy in L^1. Show there exists $g \in \mathrm{AC}(I)$ with $g = f$ a.e.

9. Suppose f, $f' \in L^1(\mathbf{R})$. Show that

   $$\int_{\mathbf{R}} f' = 0.$$