MA 527
Review problems for Exam 1 with solutions

Chapter 7: Basic linear algebra:
Rank, basis, independence, solving systems, determinants, inverse.

Chapter 8: Eigenvalues/vectors:
Finding eigenvalues/vectors/basis for eigenspace diagonalization/similarity properties

Chapter 4: Systems of differential equations:
Solution of systems - homogeneous and nonhomogeneous, critical points, types of critical points/stability

Rieview Problesms
1. True/False. No explanation necessary. Note that in mathematics, if a statement is sometimes true and sometimes false, then it is *false*.

(a) If A is diagonalizable, then its eigenvalues are distinct.
False. E.g. the matrix I is diagonal, hence diagonalizable, but all eigenvalues are 1.

(b) If all the eigenvalues of A equal 2, then there is a matrix P so that $P^{-1}AP = 2I$.
False. E.g., the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ has both eigenvalues 2 but is defective, hence not diagonalizable.

(c) If $B = C^{-1}AC$, then $B^3 = C^{-1}A^3C$.
True. Expand B^3 using the definition given, then cancel the terms CC^{-1}.

(d) If A^{-1} exists, then all the eigenvalues of A are non-zero.
True. If A^{-1} exists, then $\det(A) \neq 0$, so $\det(A - 0I) \neq 0$, so 0 is not an eigenvalue.

(e) There is some 3×3 matrix with all entries real but so that no eigenvalue is real.
False. $\det(A - \lambda I)$ has degree 3, hence has a real root, which is a real eigenvalue.

2. You are given the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 0 \\ 2 & 9 & 11 & 16 \end{bmatrix}$$

(a) Give a condition on $\vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so that $A\vec{x} = \vec{b}$ is solvable.

Form the augmented matrix $[A|\vec{b}]$, then do row reduction. The last row contains all 0 except $b+c-4a$ in the right column. So $b+c-4a=0$ is required for $A\vec{x}=\vec{b}$ to be solvable.

The echelon form of A is

$$E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -5 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Give a basis for the column space of A.

The echelon form has pivots in the first 2 columns, so the first 2 columns of A form a basis for the column space.

(c) Give a basis for the row space of A.

The first 2 rows of E form a basis for the row space of A.

(d) Give a basis for the null space of A.

Using the echelon form and backsubstitution, we get

$$\vec{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -4 \\ -8 \\ 0 \\ 5 \end{bmatrix}$$

(e) Determine all solutions to $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Since the vector on the right is the first column of A, we find a particular solution is $\vec{x}_1 = [1,0,0,0]^T$. So the general solution is $\vec{x}_1 + c_1\vec{v}_1 + c_2\vec{v}_2$, where $\vec{v}_1, \vec{v}_2$ are from part (d).

3. Determine which of the following matrices are diagonalizable.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -3 & 4 \end{bmatrix}$$

A has eigenvalues 1, 1, 2, but the geometric multiplicity of 1 is 1, so A is defective, hence not diagonalizable.

B has eigenvalues 1, 2, 3, hence has 3 linearly independent eigenvalues, hence is diagonalizable.

C has eigenvalues 1, 1, 2, but the geometric multiplicity of 1 is 1, so C is defective, hence not diagonalizable.

D has eigenvalues 1, 1, 2, and 1 has geometric multiplicity 2, so D is diagonalizable.

4. (The following three parts are unrelated to each other.)

(a) Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ be a set of vectors in $\mathbf{R}^n$, and let A be a nonsingular matrix. Prove that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is linearly independent if and only if $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_m\}$ is linearly independent.

Proof: Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is linearly independent. Then the only linear combination $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = 0$ is with all $c_j = 0$. To show $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_m\}$ is linearly independent, let $c_1A\vec{v}_1 + \cdots + c_mA\vec{v}_m$ be a linear combination that equals 0. By linearity,

$$A(c_1\vec{v}_1 + \cdots + c_m\vec{v}_m) = 0,$$

and since A is nonsingular, we can multiply both sides by A^{-1} to conclude that $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = 0$, and hence each $c_j = 0$. Hence $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_m\}$ is linearly independent.

Next suppose $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_m\}$ is linearly independent. Since A is nonsingular, A^{-1} exists and is nonsingular, so we can apply the argument just given to conclude that if we multiply each vector in this list by A^{-1}, the new set is linearly independent. Hence $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is linearly independent.

(b) Recall that $A \sim B$ means there is an invertible matrix P so that $A = P^{-1}BP$. Prove that if $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof: The assumptions imply that $A = P^{-1}BP$ and $B = Q^{-1}CQ$ for some invertible Q. Hence $A = P^{-1}Q^{-1}CQP = (QP)^{-1}C(QP)$, so $A \sim C$.

(c) Prove that the magnitude of any eigenvalue of an orthogonal matrix is 1. I.e., if $A^TA = I$ and $A\vec{x} = \lambda\vec{x}$ with $\vec{x} \neq 0$, then $|\lambda| = 1$.

Proof: Since $A\vec{x} = \lambda\vec{x}$, we have $\vec{x}^TA^TA\vec{x} = \lambda^2\vec{x}^T\vec{x}$, or $\|\vec{x}\|^2 = \lambda^2\|\vec{x}\|^2$. Since $\vec{x}$ is an eigenvector, it is not 0, so dividing by $\|\vec{x}\|^2$ gives $\lambda^2 = 1$. Hence $\lambda = \pm 1$, so $|\lambda| = 1$.

5. The matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1 + 2i$ with eigenvector $\begin{bmatrix} 5 \\ -1 + 2i \end{bmatrix}$ and $\lambda_2 = -1 - 2i$ with eigenvector $\begin{bmatrix} 5 \\ -1 - 2i \end{bmatrix}$. Determine the type and

stability of the origin for the system $dx/dt = Ax$. Find a real general solution, and sketch some trajectories in the phase plane (indicate the directions of the trajectories).

Using the notation from class, we have $a = -1$, $b = 2$, $\vec{u} = [5, -1]^T$, and $\vec{v} = [0, 2]^T$. Hence we have a spiral point that spirals in toward the origin, hence is stable and attracting. Checking $A[1, 0]^T = [-1, 0]^T$, we find the direction is clockwise. The real general solution is

$$e^{-t}[c_1((\cos 2t)\vec{u} - (\sin 2t)\vec{v}) + c_2((\cos 2t)\vec{v} + (\sin 2t)\vec{u})]$$

6. For the equation $y'' + y - y^2/2 = 0$, convert to a corresponding system, locate the critical points, and determine their types.

Using $x_1 = y$, $x_2 = y'$, the system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + \frac{x_1^2}{2} \end{aligned}$$

The critical points are $(0, 0)$ and $(2, 0)$.

At $(0, 0)$, the linear approximation matrix is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with eigenvalues $\lambda = \pm i$. Hence the corresponding linear system has a center at the origin and this is stable. In the nonlinear system, the point $(0, 0)$ may be stable, stable and attracting, or unstable depending on the nonlinear terms.

At $(2, 0)$, we use $\tilde{x}_1 = x_1 - 2$, $\tilde{x}_2 = x_2$ and get the system

$$\begin{aligned} \tilde{x}_1' &= \tilde{x}_2 \\ \tilde{x}_2' &= \tilde{x}_1 + \frac{\tilde{x}_1^2}{2} \end{aligned}$$

This has linear approximation matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with eigenvalues $\lambda = \pm 1$. Hence the corresponding linear system has a saddle at the origin. In this case, the nonlinear system has a saddle at $(2, 0)$, and this point is unstable.